1. **Problem**

A firm has the following production function:

$$F(K,L) = KL^2.$$

The price for one unit of *capital* is $p_K = 13$ and the price for one unit of *labor* is $p_L = 13$. Minimize the costs of the firm considering its production function and given a target production output of 430 units.

How high are in this case the minimal costs?

Solution

Step 1: Formulating the minimization problem.

$$\min_{K,L} C(K,L) = p_K K + p_L L$$

= 13K + 13L
subject to: $F(K,L) = Q$
 $KL^2 = 430$

Step 2: Lagrange function.

$$\mathcal{L}(K,L,\lambda) = C(K,L) - \lambda(F(K,L) - Q)$$

= 13K + 13L - \lambda(KL² - 430)

Step 3: First order conditions.

$$\frac{\partial \mathcal{L}}{\partial K} = 13 - \lambda L^2 = 0 \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial L} = 13 - 2\lambda K L^{2-1} = 0 \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(KL^2 - 430) = 0 \tag{3}$$

Step 4: Solve the system of equations for K, L, and λ . Equating Equations (1) and (2) after solving for λ gives:

$$\begin{array}{rcl} \frac{13}{L^2} & = & \frac{13}{2KL^{2-1}} \\ K & = & \frac{13}{2 \cdot 13} \cdot L^{2-(2-1)} \\ K & = & \frac{13}{26} \cdot L \end{array}$$

Substituting this in the optimization constraint gives:

$$KL^{2} = 430$$

$$\left(\frac{13}{26} \cdot L\right)L^{2} = 430$$

$$\frac{13}{26}L^{3} = 430$$

$$L = \left(\frac{26}{13} \cdot 430\right)^{\frac{1}{3}} = 9.50968541 \approx 9.51$$

$$K = \frac{13}{26} \cdot L = 4.7548427 \approx 4.75$$

The minimal costs can be obtained by substituting the optimal factor combination in the objective function:

$$C(K,L) = 13K + 13L$$

= 61.812955 + 123.62591
= 185.438865 \approx 185.44

Given the target output, the minimal costs are 185.44.

