1. **Problem**

A firm has the following production function:

$$F(K, L) = KL^3.$$

The price for one unit of *capital* is $p_K = 20$ and the price for one unit of *labor* is $p_L = 11$. Minimize the costs of the firm considering its production function and given a target production output of 730 units.

How high are in this case the minimal costs?

Solution

Step 1: Formulating the minimization problem.

$$\begin{aligned}\min_{K,L} C(K, L) &= p_K K + p_L L \\ &= 20K + 11L \\ \text{subject to:} \quad & F(K, L) = Q \\ & KL^3 = 730\end{aligned}$$

Step 2: Lagrange function.

$$\begin{aligned}\mathcal{L}(K, L, \lambda) &= C(K, L) - \lambda(F(K, L) - Q) \\ &= 20K + 11L - \lambda(KL^3 - 730)\end{aligned}$$

Step 3: First order conditions.

$$\frac{\partial \mathcal{L}}{\partial K} = 20 - \lambda L^3 = 0 \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial L} = 11 - 3\lambda KL^{3-1} = 0 \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(KL^3 - 730) = 0 \tag{3}$$

Step 4: Solve the system of equations for K, L, and λ.

Equating Equations (1) and (2) after solving for λ gives:

$$\begin{aligned}\frac{20}{L^3} &= \frac{11}{3KL^{3-1}} \\ K &= \frac{11}{3 \cdot 20} \cdot L^{3-(3-1)} \\ K &= \frac{11}{60} \cdot L\end{aligned}$$

Substituting this in the optimization constraint gives:

$$\begin{aligned}KL^3 &= 730 \\ \left(\frac{11}{60} \cdot L\right) L^3 &= 730 \\ \frac{11}{60} L^4 &= 730 \\ L &= \left(\frac{60}{11} \cdot 730\right)^{\frac{1}{4}} = 7.94365468 \approx 7.94 \\ K &= \frac{11}{60} \cdot L = 1.45633669 \approx 1.46\end{aligned}$$

The minimal costs can be obtained by substituting the optimal factor combination in the objective function:

$$\begin{aligned} C(K,L) &= 20K + 11L \\ &= 29.126734 + 87.380201 \\ &= 116.506935 \approx 116.51 \end{aligned}$$

Given the target output, the minimal costs are 116.51.